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Curious Properties of Simple Random Walks

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A simple random walker on the line of integers shows remarkable similarities to relativistic particles.

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Consider a particle performing an unrestricted random walk on the line of integers $\mathbb{Z}$ starting at the origin. Strictly speaking, one deals with an ensemble of random walkers; talking about a single one is just a convenient abbreviation. Let the probabilities of a step to the right and to the left be p and q, respectively $(p+q=1)$. After n steps, the expectation value of the particle's position is[1]

$$\langle x(n)\rangle = (p-q)n \tag{1}$$

with variance

$$\langle \Delta x^2(n)\rangle = 4pqn \tag{2}$$

and standard deviation

$$\lambda := \langle \Delta x^2(n)\rangle^{1/2} = 2(pqn)^{1/2} \tag{3}$$

Since the random walk is unrelated to any other process, the intervals τ between successive steps are most naturally (though not necessarily) assumed to be equal. The jump interval τ thus becomes the only available

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natural unit of time and it can be set equal to unity. The number of steps n then directly measures time t.

Motion to the right being taken as a positive, the drift velocity of the particle becomes

$$v = p - q \tag{4}$$

Thus, a particle performing a symmetric random walk has a drift velocity $v = 0$, and hence it is *at rest*. At the other extreme, a particle moving to the right (left) with probability one has the drift velocity $v = +1$ ($v = -1$), that is to say, of maximum absolute value. A random walk is therefore inherently associated with a limiting speed (which, by our choice of units, has magnitude 1).

In terms of the drift velocity v, the probabilities p and q are

$$p = \frac{1+v}{2}, \qquad q = \frac{1-v}{2} \tag{5}$$

and Eqs. (1–3) become, respectively

$$\langle x(n) \rangle = v \cdot n \tag{6}$$

$$\langle \Delta x^2(n) \rangle = (1 - v^2) n \tag{7}$$

$$\lambda = [(1 - v^2) n]^{1/2} \tag{8}$$

The standard deviation λ measures the extension, or "smearing out," of a particle performing a random walk. For a particle at rest,

$$\lambda = n^{1/2} =: \lambda_0 \tag{9}$$

while for a particle moving with speed v, we have

$$\lambda = (1 - v^2)^{1/2} \lambda_0 \tag{10}$$

We see that the moving particle undergoes a Lorentz contraction, which is most remarkable. We shall now investigate this more carefully.

Again, on a line consider two independent random walkers **1** and **2** with drift velocities v_1 and v_2, respectively. We only assume that the jump interval τ is the same for both walkers (or particles). How does particle **1** see particle **2**, or, in other words, what is the relative motion of **2** with respect to **1**?

At an epoch, **1** takes a step to the right (left) with probability p_1 (q_1). At the same time, **2** jumps to the right (left) with probability p_2 (q_2).

Hence, with respect to **1**, particle **2** takes a double step to the right of left with probabilities

$$p := p_{12} = q_1 p_2, \qquad q := q_{12} = p_1 q_2 \tag{11}$$

respectively. It may, however, also stay in place relative to **1**, with probability

$$r := r_{12} = p_1 p_2 + q_1 q_2 \tag{12}$$

Obviously

$$p + q + r = 1 \tag{13}$$

It is worthwhile to note that only double steps are observable. Hence, two and only two kinds of particles are possible. At a given epoch, two particles are at sites of either the same or the opposite parity (that is, with an even or an odd number of steps between them). Particles with equal parities will collide with probability one, whereas those with opposite parities will never meet (though, with probability one, they will approach each other to one step).

When both particles jump in the same direction no change is observed. It is amusing to imagine that the random walkers' notion of time is synonymous with something happening. Therefore, it is interesting to renormalize the probabilities p and q disregarding r. As a result, **1** sees **2** as performing a random walk with probabilities

$$P_{12} = \frac{p}{p+q}, \qquad Q_{12} = \frac{q}{p+q} \tag{14}$$

Since this again represents an ordinary random walk, Eq. (10) holds, and, consequently, **1** sees **2** undergoing a Lorentz contraction

$$\lambda_{12} = (1 - V_{12}^2)^{1/2} \lambda_0 \tag{15}$$

where $V_{12} = P_{12} - Q_{12}$ is the drift velocity of **2** relative to **1**.

Our reasoning is manifestly symmetric in **1** and **2**. Thus, we conclude that the behavior of simple random walkers on a line indeed mimics relativistic kinematics.

Now, we extend our calculation to include a third walker, **3**, in order to study the observation of a moving particle from two different inertial frames. We take, arbitrarily, **1** and **2** to represent *observers* attached to inertial frames, while **3** shall represent a uniformly moving test particle.

With self-explanatory notations we have

$$V_{ik}=\frac{p_k q_i - p_i q_k}{p_k q_i + p_i q_k}, \qquad i, k = 1, 2, 3 \tag{16}$$

From Eq. (16), with $(ik) = (23)$, we obtain

$$p_3=\frac{p_2(1+V_{23})}{1+V_{23}(p_2-q_2)} \tag{17}$$

$$q_3=\frac{q_2(1-V_{23})}{1+V_{23}(p_2-q_2)} \tag{18}$$

After substituting p_3, q_3 from Eqs. (17), (18) into (16) with $(ik) = (23)$ and a few elementary manipulations, we finally get

$$V_{13}=\frac{V_{12}+V_{23}}{1+V_{12}V_{23}} \tag{19}$$

which, indeed, is the relativistic addition formula for velocities.

Now, it becomes clear that **1** and **2** observe **3** in agreement with special relativity. Indeed, assume that **1** is "at rest" while **2** is comoving with **3**, that is,

$$V_{12}=V_{13}=V, \qquad V_{23}=0 \tag{20}$$

In this case, **2** finds for the extension, or standard deviation, of **3** the value λ_0, while **1** measures a smaller extension λ_{13}, given by Eq. (15), *mutatis mutandis*.

Naturally, one asks the question whether the Fitzgerald time dilatation also applies. We observe that after the same number of steps, the extension of a moving particle is smaller than that of a particle at rest. Thus, intuition tells us that the moving particle's clock runs slow. Yet, if one tries to measure the time dilatation by comparing the extensions one comes up with a wrong answer, namely, one finds a factor of $(1-v^2)$ instead of $(1-v^2)^{1/2}$.

I do not have a convincing argument for proper relativistic behavior of random walkers in this respect. Yet I can contrive, albeit rather artificially, a characteristic time with the correct velocity dependence.

Consider the waiting times n_+, n_- for particles initially moving to the right and left, respectively, to reverse their headings. It is easy to calculate their expectation values $\langle n_+ \rangle$, $\langle n_- \rangle$, variances $\langle \Delta n_+^2 \rangle$, $\langle \Delta n_-^2 \rangle$, and standard deviations θ_+, θ_-:

$$\langle n_+ \rangle = \frac{p}{q}, \qquad \langle n_- \rangle = \frac{q}{p} \tag{21}$$

$$\theta_+^2 = \langle \Delta n_+^2 \rangle = \frac{p}{q^2}, \qquad \theta_-^2 = \langle \Delta n_-^2 \rangle = \frac{q}{p^2} \tag{22}$$

Now, we define a characteristic period θ for a particle by taking the geometric mean of θ_+ and θ_-,

$$\theta := (\theta_+ \theta_-)^{1/2} = \frac{1}{(pq)^{1/2}} = \frac{2}{(1-v^2)^{1/2}} \tag{23}$$

or

$$\theta = \frac{\theta_0}{(1-v^2)^{1/2}} \tag{24}$$

with

$$\theta_0 := \theta(v=0) = 2 \tag{25}$$

We see that θ indeed undergoes a Fitzgerald time dilatation.

Thus, we see that simple random walkers on a line show a remarkable analogy to particles in the special theory of relativity. The reason for this is, of course, the existence of a limiting speed which is *a priori* built into the theory.

Unfortunately, the results do not generalize to d (>1) dimensions, since the "Lorentz contraction" turns out to be isotropic in space.

It remains to be seen whether there is some deeper principle underlying these speculations or whether they are to persist just as curiosities.

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